Introduction to the Zilber-Pink conjecture

Vahagn Aslanyan

University of East Anglia

15 April 2020

Dimension of intersection

- Given two varieties V and W in $\mathbb{C}^n$, one expects that $\dim(V \cap W) = \dim V + \dim W - n$.
- Two curves in a two-dimensional space are likely to intersect, while two curves in a three-dimensional space are not. If they do intersect, then we have an *unlikely intersection*.

Theorem

Let $V, W \subseteq \mathbb{C}^n$ be irreducible algebraic varieties and $X \subseteq V \cap W$ be an irreducible component of the intersection. Then

$$\dim X \geq \dim V + \dim W - n.$$

Definition

X is an *atypical* component of $V \cap W$ if $\dim X > \dim V + \dim W - n$.

Algebraic tori

- Let $\mathbb{G}_{\mathrm{m}}(\mathbb{C})$ be the multiplicative group $(\mathbb{C}^{\times}; \cdot, 1)$.
- An *algebraic torus* is an irreducible algebraic subgroup of $\mathbb{G}_{\mathrm{m}}^{n}(\mathbb{C})$.
- A torus of dimension d is isomorphic to $\mathbb{G}_{\mathrm{m}}^{d}(\mathbb{C})$.
- Algebraic subgroups of $\mathbb{G}_{\mathrm{m}}^{n}(\mathbb{C})$ are defined by several equations of the form
$$y_1^{m_1} \cdots y_n^{m_n} = 1.$$
- For any such subgroup the connected component of the identity element is an irreducible algebraic subgroup of finite index and is a torus. Every such group is equal to a disjoint union of a torus and its torsion cosets.

Special and atypical subvarieties

Definition

Irreducible components of algebraic subgroups of $\mathbb{G}_m^n(\mathbb{C})$, that is, torsion cosets of tori, are the *special* varieties. These are defined by equations of the form $y_1^{m_1} \cdots y_n^{m_n} = \zeta$ where ζ is a root of unity.
If $U \subseteq \mathbb{C}^n$ is a rational translate of a $\mathbb{Q}$-linear subspace then $\exp(2\pi i U)$ is special.

Definition

For a variety $V \subseteq \mathbb{G}_m^n(\mathbb{C})$ and a special variety $S \subseteq \mathbb{G}_m^n(\mathbb{C})$, a component X of the intersection $V \cap S$ is an *atypical subvariety* of V if $\dim X > \dim V + \dim S - n$.

Definition

The *atypical set* of V, denoted $\mathrm{Atyp}(V)$, is the union of all atypical subvarieties of V.

Conjecture on Intersections with Tori

Conjecture (CIT)

Every algebraic variety in $\mathbb{G}_m^n(\mathbb{C})$ *contains only finitely many maximal atypical subvarieties.*

Conjecture (CIT)

Let $V \subseteq \mathbb{G}_m^n(\mathbb{C})$ *be an algebraic variety. Then there is a finite collection* Σ *of proper special subvarieties of* $\mathbb{G}_m^n(\mathbb{C})$ *such that every atypical subvariety* X *of* V *is contained in some* $T \in \Sigma$.

Conjecture (CIT)

Let $V \subseteq \mathbb{G}_m^n(\mathbb{C})$ *be an algebraic variety. Then* $\mathrm{Atyp}(V)$ *is a Zariski closed subset of* V.
If V *is not contained in a proper special subvariety of* $\mathbb{G}_m^n(\mathbb{C})$ *then* $\mathrm{Atyp}(V)$ *is a proper Zariski closed subset of* V.

A brief history of CIT

- In his model theoretic analysis of the complex exponential field and Schanuel's conjecture, Zilber came up with CIT [Zil02].
- Schanuel's conjecture (see [Lan66, p. 30]) states that for any $\mathbb{Q}$-linearly independent complex numbers $z_1, \ldots, z_n$

  $$\operatorname{td}_{\mathbb{Q}} \mathbb{Q}(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}) \geq n.$$

  Assuming CIT, Schanuel's conjecture implies a uniform version of itself.
- Zilber showed that the generalisation of CIT to semi-abelian varieties implies the Manin-Mumford and Mordell-Lang conjectures.

A brief history of CIT

- Bombieri-Masser-Zannier independently proposed an equivalent conjecture in [BMZ07].
- They had proven CIT for curves in an earlier paper [BMZ99].
- Pink proposed a similar and more general conjecture for mixed Shimura varieties, again independently [Pin05b, Pin05a]. which generalises André-Oort, Manin-Mumford and Mordell-Lang.
- The general conjecture is now known as the Zilber-Pink conjecture.
- We will only consider the Zilber-Pink conjecture for semi-abelian varieties and $Y(1)^n$.

Special and atypical varieties in the semi-abelian setting

Definition

- An *abelian variety* is a connected complete algebraic group (think of elliptic curves).
- A *semi-abelian variety* is a commutative algebraic group $\mathfrak{S}$ which is an extension of an abelian variety by a torus. For example, a product of elliptic curves and algebraic tori is a semi-abelian variety.

Definition

- A *special* subvariety of a semi-abelian variety $\mathfrak{S}$ is a torsion coset of a semi-abelian subvariety of $\mathfrak{S}$.
- Let $\mathfrak{S}$ be a semi-abelian variety and $V \subseteq \mathfrak{S}$ be an algebraic subvariety. An atypical subvariety of V in $\mathfrak{S}$ is a component X of an intersection of V with a special variety $T \subseteq \mathfrak{S}$ such that $\dim X > \dim V + \dim T - \dim \mathfrak{S}$.

ZP for semi-abelian varieties

Conjecture (Zilber–Pink for semi-abelian varieties)

Let $\mathfrak{S}$ be a semi-abelian variety and $V \subseteq \mathfrak{S}$ be an algebraic subvariety. Then V contains only finitely many maximal atypical subvarieties.

Conjecture

Let $\mathfrak{S}$ be a semi-abelian variety and $V \subseteq \mathfrak{S}$ be an algebraic subvariety. Then there is a finite collection Σ of proper special subvarieties of $\mathfrak{S}$ such that every atypical subvariety X of V is contained in some $T \in \Sigma$.

Conjecture

Let $V \subseteq \mathfrak{S}$ be an algebraic variety. Then $\mathrm{Atyp}(V)$ is a Zariski closed subset of V.
If V is not contained in a proper special subvariety of $\mathfrak{S}$ then $\mathrm{Atyp}(V)$ is a proper Zariski closed subset of V.

Manin-Mumford conjecture

Theorem (Manin-Mumford conjecture; Raynaud, Hindry)

Let $\mathfrak{S}$ be a semi-abelian variety and $V \subsetneq \mathfrak{S}$ be a subvariety. Then V contains only finitely many maximal special subvarieties. In particular, an irreducible curve contains finitely many special points unless it is special itself.

Remark

Lang asked the following question in the 1960s. Assume $f(x, y) = 0$ contains infinitely many points (ξ, η) whose coordinates are roots of unity. What can be said about f?

Zilber-Pink implies Manin-Mumford

The Manin-Mumford conjecture can be deduced from Zilber-Pink.

- First, we may assume V is not contained in a proper special subvariety of $\mathfrak{S}$. Otherwise we would replace $\mathfrak{S}$ by the smallest special subvariety containing V and translate by a torsion point if necessary. This is to make sure that V is not an atypical subvariety of V.
- Now if $T \subseteq V \subsetneq \mathfrak{S}$ and T is special then it is an atypical subvariety of V for
$$\dim T > \dim V + \dim T - \dim \mathfrak{S}.$$
- If $T \subseteq V$ is maximal special then either T is maximal atypical in V or it is contained (and is maximal special) in a maximal atypical subvariety of V. So we can proceed inductively.

Weakly special and Γ-special subvarieties in semi-abelian varieties

Definition

Let $\mathfrak{S}$ be a semi-abelian variety and let $\Gamma \subseteq \mathfrak{S}$ be a subgroup of finite rank.

- A *weakly special* subvariety of $\mathfrak{S}$ is a coset of an irreducible algebraic subgroup.
- A Γ-*special* subvariety of $\mathfrak{S}$ is a translate of an irreducible algebraic subgroup by a point of Γ. In other words, a weakly special subvariety is Γ-special if it contains a point of Γ.

Mordell-Lang

Theorem (Mordell-Lang conjecture; Faltings, Vojta, McQuillan,...)

Let $\mathfrak{S}$ be a semi-abelian variety and let $\Gamma \subseteq \mathfrak{S}$ be a subgroup of finite rank. Then an algebraic variety $V \subseteq \mathfrak{S}$ contains only finitely many maximal Γ-special subvarieties.

Theorem (Mordell-Lang conjecture)

If $V \cap \Gamma$ is Zariski dense in V then V is a finite union of Γ-special varieties.

Remark

The Mordell-Lang conjecture for abelian varieties, combined with the Mordell-Weil theorem, implies the Mordell conjecture (Faltings's theorem), namely, a curve of genus ≥ 2 defined over $\mathbb{Q}$ has only finitely many rational points.

Weak ZP for semi-abelian varieties

Theorem (Zilber, Kirby, Bombieri-Masser-Zannier)

Let V be an algebraic subvariety of a semi-abelian variety $\mathfrak{S}$. Then there is a finite collection Σ of proper algebraic subgroups of $\mathfrak{S}$ such that every atypical component of an intersection of V with a weakly special subvariety of $\mathfrak{S}$ is contained in a coset of some $T \in \Sigma$.

This theorem is also true uniformly for parametric families of algebraic varieties. The proof is based on the Ax-Schanuel theorem.

Theorem (Ax, 1971)

If $f_1(\bar{z}), \ldots, f_n(\bar{z})$ are complex analytic functions defined on some open domain $U \subseteq \mathbb{C}^m$, and no $\mathbb{Q}$-linear combination of f_i's is constant, then

$$\operatorname{td}_{\mathbb{Q}}(f_1, \ldots, f_n, e^{f_1}, \ldots, e^{f_n}) \geq n + \operatorname{rk}\left(\frac{\partial f_i}{\partial z_j}\right).$$

The j-function

- Let $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the complex upper half-plane.
- $\mathrm{GL}_2^+(\mathbb{R})$ is the group of 2×2 matrices with real entries and positive determinant. It acts on $\mathbb{H}$ via linear fractional transformations. That is, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$ we define

$$gz = \frac{az+b}{cz+d}.$$

- The function $j : \mathbb{H} \to \mathbb{C}$ is a modular function of weight 0 for the modular group $\mathrm{SL}_2(\mathbb{Z})$ defined and analytic on $\mathbb{H}$.
- $j(\gamma z) = j(z)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Modular polynomials

- For $g \in \mathrm{GL}_2^+(\mathbb{Q})$ we let $N(g)$ be the determinant of g scaled so that it has relatively prime integral entries.
- For each positive integer N there is an irreducible polynomial $\Phi_N(X, Y) \in \mathbb{Z}[X, Y]$ such that whenever $g \in \mathrm{GL}_2^+(\mathbb{Q})$ with $N = N(g)$, the function $\Phi_N(j(z), j(gz))$ is identically zero.
- Conversely, if $\Phi_N(j(x), j(y)) = 0$ for some $x, y \in \mathbb{H}$ then $y = gx$ for some $g \in \mathrm{GL}_2^+(\mathbb{Q})$ with $N = N(g)$.
- The polynomials Φ_N are called *modular polynomials*.
- $\Phi_1(X, Y) = X - Y$ and all the other modular polynomials are symmetric.
- For a complex number w its *Hecke orbit* is the set $\{z \in \mathbb{C} : \Phi_N(w, z) = 0 \text{ for some } N\}$.

Special and atypical varieties in the modular setting

Definition

A *special* subvariety of $\mathbb{C}^n$ (coordinatised by $\bar{y}$) is an irreducible component of a variety defined by modular equations, i.e. equations of the form $\Phi_N(y_i, y_k) = 0$ for some $1 \leq i, k \leq n$ where $\Phi_N(X, Y)$ is a modular polynomial.

Definition

A subvariety $U \subseteq \mathbb{H}^n$ (i.e. an intersection of $\mathbb{H}^n$ with some algebraic variety) is called $\mathbb{H}$-*special* if it is defined by some equations of the form $z_i = g_{i,k} z_k,\ i \neq k$, with $g_{i,k} \in \mathrm{GL}_2^+(\mathbb{Q})$, and some equations of the form $z_i = \tau_i$ where $\tau_i \in \mathbb{H}$ is a quadratic number. For such a U the image $j(U)$ is special.

Atypical subvarieties and $\mathrm{Atyp}(V)$ are defined exactly as before.

Modular Zilber-Pink

Conjecture

Every algebraic variety in $\mathbb{C}^n$ contains only finitely many maximal atypical subvarieties.

Conjecture

Let $V \subseteq \mathbb{C}^n$ be an algebraic variety. Then there is a finite collection Σ of proper special subvarieties of $\mathbb{C}^n$ such that every atypical subvariety X of V is contained in some $T \in \Sigma$.

Conjecture

Let $V \subseteq \mathbb{C}^n$ be an algebraic variety. Then $\mathrm{Atyp}(V)$ is a Zariski closed subset of V.
If V is not contained in a proper special subvariety of $\mathbb{C}^n$ then $\mathrm{Atyp}(V)$ is a proper Zariski closed subset of V.

André-Oort conjecture for $Y(1)^n$

Theorem (Pila)

Let $V \subsetneq \mathbb{C}^n$ be a variety. Then V contains only finitely many maximal special subvarieties.

Remark

This theorem follows from modular ZP.

Weakly special and Γ-special varieties in $Y(1)^n$

Definition

- A *weakly special* subvariety of $\mathbb{C}^n$ is an irreducible component of a variety defined by equations of the form $\Phi_N(x_i, x_k) = 0$ and $x_l = c_l$ where $c_l \in \mathbb{C}$ is a constant.
- A special variety is called *strongly* special if no coordinate is contant on it.

Definition

Let Γ be a finite subset of $\mathbb{C}$.

- A point $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ is Γ-special if every coordinate of z is either special or is in the Hecke orbit of some $\gamma \in \Gamma$.
- A weakly special subvariety of $\mathbb{C}^n$ is Γ-*special* if it contains a Γ-special point.

Modular Mordell-Lang

Theorem (Habegger-Pila, [HP12])

Let $V \subseteq \mathbb{C}^n$ be an algebraic variety and let $\Gamma \subseteq \mathbb{Q}^{\mathrm{alg}}$ be a finite subset. Then V contains only finitely many maximal Γ-special subvarieties.

Weak modular ZP

Definition

An atypical subvariety of V is called *strongly atypical* if it does not have any constant coordinates.

Theorem (Weak Modular Zilber-Pink, [PT16])

Every algebraic subvariety $V \subseteq \mathbb{C}^n$ contains only finitely many maximal strongly atypical subvarieties.

Weak ZP is true uniformly in parametric families.

Theorem (Uniform weak modular ZP)

Given a parametric family of algebraic subvarieties $(V_q)_{q \in Q}$ of $\mathbb{C}^n$, there is a finite collection Σ of proper special subvarieties of $\mathbb{C}^n$ such that for every $q \in Q$ and for every strongly atypical subvariety X of V_q there is $T \in \Sigma$ with $X \subseteq T$.

Ax-Schanuel for the j-function

- Let $\Gamma := \{(\bar{z}, j(\bar{z})) : z_i \in \mathbb{H}\} \subseteq \mathbb{C}^{2n}$ be the graph of j.
- Let $\mathrm{pr}_j : \mathbb{C}^{2n} \to \mathbb{C}^n$ be the projection onto the j-coordinates, i.e. the second n coordinates.

Theorem (Pila-Tsimerman, [PT13])

Let $V \subseteq \mathbb{C}^{2n}$ be an algebraic variety and let A be an analytic component of the intersection $V \cap \Gamma$. If $\dim A > \dim V - n$ then $\mathrm{pr}_j A$ is contained in a proper weakly special subvariety of $\mathbb{C}^n$.

Theorem (Uniform Ax-Schanuel for j, [Asl18, Theorem 7.8])

Let $(V_q)_{q \in Q}$ be a parametric family of algebraic subvarieties of $\mathbb{C}^{2n}$. Then there is a finite collection Σ of proper special subvarieties of $\mathbb{C}^n$ such that for every $q \in Q(\mathbb{C})$, if A_q is an analytic component of the intersection $V_q \cap \Gamma$ with $\dim A_q > \dim V_q - n$, and no coordinate is constant on $\mathrm{pr}_j A_q$, then $\mathrm{pr}_j A_q$ is contained in some $T \in \Sigma$.

Proof of uniform weak ZP

- Fix a $q \in Q(\mathbb{C})$ and consider the variety V_q.
- Let S be special and $X \subseteq V_q \cap S$ be strongly atypical, i.e. $\dim X > \dim V_q + \dim S - n$.
- Let $U \subseteq \mathbb{H}^n$ be special such that $j(U) = S$.
- $\dim(U \times X) \cap \Gamma = \dim X > \dim(U \times V_q) - n$.
- We can now apply uniform Ax-Schanuel to the family $W_r \times V_q$ where W_r varies over all subvarieties of $\mathbb{C}^n$ defined by $\mathrm{GL}_2(\mathbb{C})$-relations.
- A differential algebraic proof is given in [Asl18] (Theorem 5.2).

Optimal varieties

Let $\mathfrak{S}$ be a semi-abelian variety or $Y(1)^n$.

Definition

For $X \subseteq \mathfrak{S}$ the *special closure* of X, denoted $\langle X \rangle$, is the smallest special variety containing X.

Definition

- For a subvariety $X \subseteq \mathfrak{S}$ the *defect* of X is the number

$$\delta(X) := \dim\langle X \rangle - \dim X.$$

- Let V be a subvariety of $\mathfrak{S}$. A subvariety $X \subseteq V$ is *optimal* (in V) if for every subvariety $Y \subseteq V$ with $X \subsetneq Y$ we have $\delta(Y) > \delta(X)$.

Remark

It is easy to show that a maximal atypical subvariety is optimal.

Zilber-Pink in terms of optimal varieties

The following conjecture is equivalent to Zilber-Pink.

Conjecture

Let V be a subvariety of $\mathfrak{S}$. Then V contains only finitely many optimal subvarieties.

Daw and Ren reduced ZP to a point counting problem.

Conjecture

Let V be a subvariety of $\mathfrak{S}$. Then V contains only finitely many points which are optimal in V.

Theorem ([DR18])

The above conjecture implies ZP.

o-minimality proof of weak ZP (sketch)

- We need to show that V contains finitely many optimal subvarieties with no constant coordinate.
- Restrict j to a fundamental domain $\mathbb{F}$. Then it is definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$. For $A \subseteq \mathbb{F}^n$ let $\langle A \rangle$ be the smallest special variety containing A, and define $\delta(A) = \dim\langle A \rangle - \dim A$.
- Let $Z := j^{-1}(V) \cap \mathbb{F}^n$.
- If $U \subseteq \mathbb{F}^n$ is special and a component $X \subseteq j(U) \cap V$ is optimal in V, then $A := j^{-1}(X) \subseteq U \cap Z$ is optimal in Z.
- Consider the set $\mathfrak{M}$ of all Mobius subvarieties (i.e. defined by $\mathrm{SL}_2(\mathbb{R})$-relations) M of $\mathbb{F}^n$ such that $\dim M - \dim(M \cap Z) < \dim N - \dim(N \cap Z)$ whenever $M \cap Z \subsetneq N \cap Z$ and $M \cap Z$ has no constant coordinate. This is a definable set.
- Ax-Schanuel theorem implies that $\mathfrak{M}$ consists of strongly special subvarieties of $\mathbb{F}^n$, that is, subvarieties defined by $\mathrm{GL}_2^+(\mathbb{Q})$-relations.
- Thus, we have a definable subset of a countable set in an o-minimal structure which must be finite.

Yet another formulation of ZP

Let $\mathfrak{S}$ be a semi-abelian variety or $Y(1)^n$.
For an integer d let $\mathfrak{S}^{[d]}$ denote the union of all special subvarieties of $\mathfrak{S}$ of dimension $\leq d$.

Conjecture

Let $V \subseteq \mathfrak{S}$ be an algebraic variety which is not contained in a proper special subvariety of $\mathfrak{S}$. Then $V \cap \mathfrak{S}^{[\dim \mathfrak{S} - \dim V - 1]}$ is not Zariski dense in V.

Further known cases and reductions

- CIT for curves: Bombieri-Masser-Zannier [BMZ99].
- ZP for curves in abelian varieties defined over a number field: Habegger and Pila [HP16].
- ZP for *non-degenerate* varieties in $\mathbb{G}_{\mathrm{m}}^n$ defined over $\mathbb{Q}^{\mathrm{alg}}$.
- Habegger and Pila reduced ZP in the abelian and modular settings to a "Large Galois Orbit" statement [HP16].
- Pila and Scanlon have established a differential algebraic ZP theorem where they allow atypical subvarieties to have constant coordinates which are non-constant in the differential field [Sca18].
- A weak ZP statement in the modular setting where atypical subvarieties are allowed to have constant coordinates which are special was proven in [Asl19]. More general results, combining weak ZP with Mordell-Lang, have also been proven there.
- See [Zan12] for various other statements.

J-special varieties

Consider the function $J : \mathbb{H} \to \mathbb{C}^3,\ J : z \mapsto (j(z), j'(z), j''(z))$.
Recall that a subvariety $U \subseteq \mathbb{H}^n$ is called $\mathbb{H}$-*special* if it is defined by some equations of the form $z_i = g_{i,k} z_k,\ i \neq k$, with $g_{i,k} \in \mathrm{GL}_2^+(\mathbb{Q})$, and some equations of the form $z_i = \tau_i$ where $\tau_i \in \mathbb{H}$ is a quadratic number. For such a U we denote by $\langle\langle U \rangle\rangle$ the Zariski closure of $J(U)$ over $\mathbb{Q}^{\mathrm{alg}}$.

Definition

A *J-special* subvariety of $\mathbb{C}^{3n}$ is a set $\langle\langle U \rangle\rangle$ where U is a special subvariety of $\mathbb{H}^n$.

Definition

For a variety $V \subseteq \mathbb{C}^{3n}$ we let the *J-atypical set* of V, denoted $\mathrm{Atyp}_J(V)$, be the union of all atypical components of intersections $V \cap T$ in $\mathbb{C}^{3n}$ where $T \subseteq \mathbb{C}^{3n}$ is a J-special variety.

Modular ZP with Derivatives

In unpublished notes Pila proposed the following conjecture.

Conjecture (Pila, "MZPD")

For every algebraic variety $V \subseteq \mathbb{C}^{3n}$ there is a finite collection Σ of proper $\mathbb{H}$-special subvarieties of $\mathbb{H}^n$ such that

$$\mathrm{Atyp}_J(V) \cap J(\mathbb{H}^n) \subseteq \bigcup_{\substack{U \in \Sigma \\ \bar{\gamma} \in \mathrm{SL}_2(\mathbb{Z})^n}} \langle\langle \bar{\gamma} U \rangle\rangle.$$

Weak versions and differential/functional analogues of this conjecture have been proven in [Spe19] and [Asl18].

For example, the above statement holds if we replace $\mathrm{Atyp}_J(V)$ with the *strongly* J-atypical set of V which is the union of all J-atypical subvarieties X of V such that none of the irreducible components of $X \cap J(\mathbb{H}^n)$ has a constant coordinate.

Vahagn Aslanyan.
Weak Modular Zilber-Pink with Derivatives.
Preprint, arXiv:1803.05895, 2018.

Vahagn Aslanyan.
Some remarks on atypical intersections.
Preprint, arXiv:1905.00827, 2019.

Enrico Bombieri, David Masser, and Umberto Zannier.
Intersecting a curve with algebraic subgroups of multiplicative groups.
IMRN, 20:1119–1140, 1999.

Enrico Bombieri, David Masser, and Umberto Zannier.
Anomalous subvarieties - structure theorems and applications.
IMRN, 19, 2007.

Christopher Daw and Jinbo Ren.
Applications of the hyperbolic Ax-Schanuel conjecture.
Compositio Mathematica, 154(9):1843–1888, 2018.

Philipp Habegger and Jonathan Pila.

Some unlikely intersections beyond André-Oort.
Compos. Math., 148(1):1–27, 2012.

Philipp Habegger and Jonathan Pila.
O-minimality and certain atypical intersections.
Ann. Sci. Éc. Norm. Supér., 49(4):813–858, 2016.

Serge Lang.
Introduction to Transcendental Numbers.
Addison-Wesley, Berlin, Springer-Verlag, 1966.

Richard Pink.
A combination of the conjectures of Mordell-Lang and André-Oort.
In F. Bogomolov and Y. Tschinkel, editors, *Geometric methods in algebra and number theory*, volume 235, pages 251–282. Progress in Mathematics, Birkhäuser Boston, 2005.

Richard Pink.
A common generalization of the conjectures of André-Oort, Manin-Mumford and Mordell-Lang.

Available at
https://people.math.ethz.ch/~pink/ftp/AOMMML.pdf, 2005.

Jonathan Pila and Jacob Tsimerman.
Ax-Lindemann for A_g.
Accepted for publication in the Annals of Mathematics, 2013.

Jonathan Pila and Jacob Tsimerman.
Ax-Schanuel for the j-function.
Duke Math. J., 165(13):2587–2605, 2016.

Thomas Scanlon.
Differential algebraic Zilber-Pink theorems.
Available at math.berkeley.edu/~scanlon/papers/ZP-Oxford-July-2018.pdf, 2018.

Haden Spence.
A modular André-Oort statement with derivatives.
Proceedings of the Edinburgh Mathematical Society, 62(2):323–365, 2019.

Umberto Zannier.

Some problems of unlikely intersections in arithmetic and geometry, volume 181 of *Annals of Mathematics Studies*.
Princeton University Press, Princeton, NJ, 2012.
With appendixes by David Masser.

Boris Zilber.
Exponential sums equations and the Schanuel conjecture.
J.L.M.S., 65(2):27–44, 2002.